

Andrew Snowden - Jacquet-Langlands

1. Statement of J-L

If G is a t.d. l.c. group (finite, profinite, GCF)

If G is cpt, irred admissible reps are fin. dim.

If $G = \mathrm{GL}(n, F)$, then fin. dim + irred. admissible \Rightarrow one dim ℓ .

So every thing is infinite dim ℓ .

If π is a rep^h of G on V irred admissible,

and if $\phi: G \rightarrow \mathbb{C}$ is locally constant compactly supported,

$$\text{then } \pi(\phi)v = \int_G \phi(g)\pi(g)v dg.$$

$\pi(\phi)$ finite rank since π admissible

The character of π is the distribution χ_{π}

$$\text{given by } \chi_{\pi}(\phi) := \mathrm{tr}(\pi(\phi))$$

Thm (Harish-Chandra): χ_{π} is represented by a

function that is continuous on the regular elements.

(call the function $\tilde{\chi}_{\pi}$)

$$\tilde{\chi}_{\pi}(\phi)$$

Jacquet - Langlands: let F/\mathbb{Q}_p finite extension.

let B/F be a division algebra of rank n ,

$$G' := B^\times. \quad \text{---}$$

$\left\{ \begin{array}{l} \text{The irreducible admissible repr's of } G' \\ \text{bijectively} \end{array} \right\}$

$\left\{ \begin{array}{l} \text{(almost) square-integrable irreducible admissible} \\ \text{repr's of } G \end{array} \right\}$

$$\tilde{\pi}' \in \widehat{G'} \quad \longleftrightarrow \quad \tilde{\pi} \in \widehat{G}$$

iff

$$(-1)^{n-1} \chi_{\tilde{\pi}}(g) = \chi_{\tilde{\pi}'}(g) \quad \forall g \text{ elliptic semi-simple in } G$$

so then g corresponds to some conjugacy class

in $\mathfrak{B}(G')$

This correspondence preserves L and ϵ -factors up to some sign ± 1 .

Global variant of J-L : F/\mathbb{Q} = finite extnsn,

$G = GL_n(A_F)$. B/F = division algebra,

$G' = (B \otimes A_F)^\times$. Then

$$\left\{ \begin{array}{l} \text{automorphic reps} \\ \text{of } G' \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{automorphic} \\ \text{reps of } G \end{array} \right\}$$

2. Applications :

Local case: Representations of division algebras
are finite dimensional. Those of $GL(n)$ are
infinite dim'l.

Global: ($n=2$) If B = quaternion algebra
is non-split at ∞

then

Jacquet-Langlands map
 \downarrow

$$\Sigma(\Gamma_0(N)) \xrightarrow{\sim} \text{Functions}\left(B^\times \backslash (B \otimes A_F^\times)^X / \mathcal{U}_A^\times \right)$$

functions
factoring through
norm

3. History:

- First ~~proof~~ proved by Jacquet - Langlands in

their book for $n=2$ (local and global cases)

- Local proof - used Weil repn
- Global proof - used trace formula

- Proven for $GL(n)$ arbitrary n

Deligne - Ichardan - Vigneras, ~~Rogawski~~

- Global proof by trace formula Rogawski
- Local proof by embedding into global case

- Construction of Drinfeld / Deligne (see book of Harris / Taylor):

Take a 1-dim'l height σ formal \mathcal{O}_F -module over the residue field. Unique, End = division algebra with Hasse invariant γ_σ . Take universal deformation

\Rightarrow formal group $/F$, Drinfeld level p^∞ structure on this. Take cohomology after you base change to \bar{F}

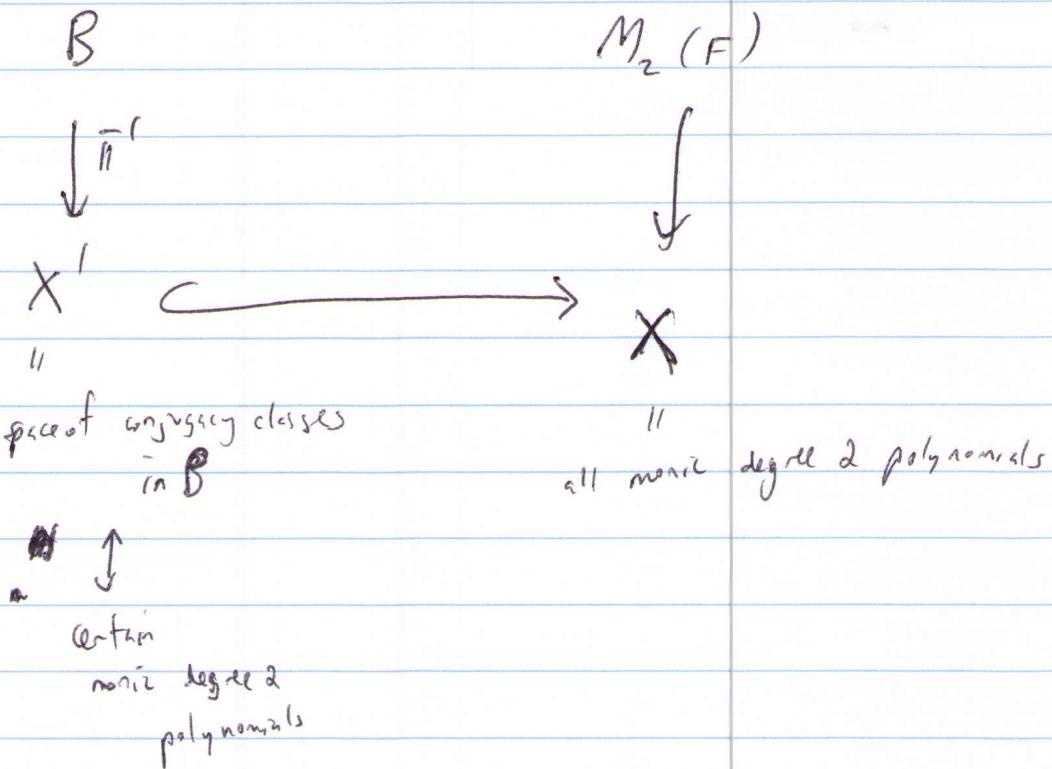
This has an action of $W_F \times GL(n, F) \times B$

$|$ acts on level structure thing that
comes from base change \backslash was deformed

4. Local proof in $GL(2)$ case (Snowden):

Set-up: Let \mathbb{F}/\mathbb{Q}_p finite.

There's a unique non-split quaternion algebra over \mathbb{F} ,
call it B .



WTS: restriction from functions on X to functions on X'
takes characters to characters

Supercuspidal repr are automatically square-integrable??

Idea: Construct some algebra A which acts
on Schwarz space of each side,

~~$\mathcal{S}(B)$ and $\mathcal{S}(M(F))$~~

$\mathcal{S}(X)$ and $\mathcal{S}(X')$

Then I'm going to show that the
 A -module structure determines which
functions are characters. Then we'll
show that the restriction map is A -linear

① A and its action on ~~$\mathcal{S}(B) \otimes \mathcal{S}(M(F))$~~
 $\mathcal{S}(X), \mathcal{S}(X')$

$A :=$ polynomial ring in the symbols $A_{\psi, v}$

where $\psi \in \hat{F}$, $v \in \hat{F}^{\times}$.

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The action is Fourier transform :

$$A_{\psi, v} \phi = v \sum_{\psi} (1 \cdot 1^{-1} v \phi^v)$$

$$\phi^v(g) = \phi(g^{-1})$$

for $\phi \in \mathcal{S}(X')$

~~To define Fourier transform on $\mathcal{J}(X')$.~~

② A -module structure on $\mathcal{J}(X')$

Thm: $\mathcal{J}(X')$ is a direct sum of simple modules,
• no two of them are isomorphic.

The simple constituents are in bijection
with unramified twists of irreducible reps.

Sketch of proof :

- For a character χ of B^* and some

$\phi \in \mathcal{J}(B)$, define

$$Z(\phi, s, \chi) = \int_{B^X} |x|^{s+1/2} \chi(g) \phi(g) dg$$

Assume $\deg(\chi) > 1$, so no L-factors

The functional equation says

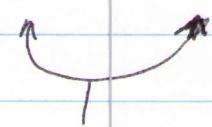
$$Z(1-s, \hat{\phi}, \chi^\vee) = (\varepsilon\text{-factor}) Z(s, \phi, \chi).$$

You can translate this into:

$$\mathcal{F}_\psi(x) = |\cdot|^2 c_\psi(x) x^v$$

↑ ↑
 Fourier transform norm

Look at functions $\phi_{x,K} =$ function x restricted to valuation K coset of B^\times and then extend ~~to~~ by \oplus to B .

$$\text{Then } A_{\psi,v} \phi_{x,K} = c_\psi(v^{-1}x) \phi_{x, K - 2m(\psi) - n(v^{-1}x)}$$


conductors

But then $\bigoplus_{K \in \mathbb{Z}^I} \phi_{x,K}$ is stable under A ,

where x is fixed.

This proves everything except "multiplicity free" statement.

③

Comparison of A -module structure

~ Enough to show \exists_{ψ} on $\mathcal{J}(x')$ coming from
 $M_2(P)$ and B are the same.